A brief tour of the Local Langlands Correspondence

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I. The local Langlands correspondence: my perspective

The local Langlands correspondence predicts:

$$\left\{ \text{reps of } p\text{-adic groups} \right\}$$

Local Galois theory $\leftrightsquigarrow$ Complex Lie theory

Where does the LLC come from?

Local class field theory (Hasse, 1930)

There is an explicit bijection

$$\left\{\begin{array}{c}\text{Finite abelian}\\ \text{extensions } L/\mathbb{Q}_p\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{Finite-index}\\ \text{subgroups of } \mathbb{Q}_p^\times\end{array}\right\}$$

$$L \longmapsto N(L^\times)$$

norm.

such that

$$\mathrm{Gal}(L/\mathbb{Q}_p) \simeq \mathbb{Q}_p^\times / N(L^\times).$$

What about nonabelian exts?

From CFT to LLC

$\mathcal{W}$ the Weil group of $\mathbb{Q}_p$ $\quad W \subseteq Gal(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$

Local class field theory gives a map

$$\left\{\begin{array}{c}\text{smooth irred representations}\\ \text{of } \boxed{\mathrm{GL}_1(\mathbb{Q}_p)}\end{array}\right\} \longrightarrow \left\{\begin{array}{c}\text{Characters}\\ \mathcal{W} \to \boxed{\mathbb{C}^\times}\end{array}\right\}$$

replace GL_1 with other grps.

replace with other complex Lie groups

Idea of how?

$L/\mathbb{Q}_p$

$$\mathbb{Q}_p^\times / N(L^\times) \simeq Gal(L/\mathbb{Q}_p)$$

$$\Rightarrow \quad \left\{\begin{array}{c}\text{reps of}\\ \mathbb{Q}_p^\times/N(L^\times)\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{reps of}\\ Gal(L/\mathbb{Q}_p)\end{array}\right\}$$

$$\left\{\begin{array}{c}\text{reps of}\\ \mathbb{Q}_p^\times\end{array}\right\} \qquad\qquad W \longrightarrow\!\!\!\!\to Gal(L/\mathbb{Q}_p) \longrightarrow \mathbb{C}^\times$$

The LLC, more explicitly

$\mathbf{G}$ split, simple connected reductive group over $\mathbb{Q}_p$ ex $\mathbf{G} = SL_n$, PGL_n, SO_n, G_2, E_8

$G = \mathbf{G}(\mathbb{Q}_p)$

$G^\vee$ dual group of $\mathbf{G}$ ($\mathbb{C}$-points)

ex $G = SL_n(\mathbb{Q}_p) \Rightarrow G^\vee = PGL_n(\mathbb{C})$

$G = SO_{2n+1}(\mathbb{Q}_p) \Rightarrow G^\vee = Sp_{2n}(\mathbb{C})$

The LLC predicts an explicit map

$$\left\{\begin{array}{c}\text{smooth irred representations}\\ \text{of } G\end{array}\right\} \longrightarrow \left\{\begin{array}{c}\text{Langlands parameters}\\ \varphi : \mathcal{W} \times \mathrm{SL}_2(\mathbb{C}) \to G^\vee\end{array}\right\}$$

satisfying certain properties.

II. Depth

Suppose $\varphi(W)$ is finite.

$\Rightarrow$ φ determines a finite Galois ext $L_\varphi/\mathbb{Q}_p$.

with $\mathrm{Gal}(L_\varphi/\mathbb{Q}_p) \simeq \varphi(W)$.

$$\begin{tikzcd}
L_\varphi \arrow[d, dash, "\text{``wildly ramified'' Gal. grp is a p-group.}"] \arrow[ddd, dash, bend right] \\
(L_\varphi)^{tr} \text{ ``tamely ramified''} \arrow[d, dash, "\text{cyclic, order prime to } p."] \\
(L_\varphi)^{un} \text{ ``unramified''} \arrow[d, dash, "\text{cyclic}"] \\
\mathbb{Q}_p
\end{tikzcd}$$

$$\left\{\begin{matrix}\text{smooth irred. representations}\\ \text{of } G\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{Langlands parameters}\\ \varphi : \mathcal{W} \times \mathrm{SL}_2(\mathbb{C}) \to G^\vee\end{matrix}\right\}$$

most well understood

$$\{\text{unipotent rep}\} \xrightarrow{\text{LLCun}} \{\varphi \text{ s.t. } L\varphi = (L\varphi)^{un} \text{ “unramified”}\}$$

$$\cap\!\mid \qquad\qquad\qquad\qquad \cap\!\mid$$

$$\{\text{depth-zero reps}\} \longrightarrow \{\varphi \text{ s.t. } L\varphi = (L\varphi)^{tr}\}$$

$$\{\text{pos.-depth rep}\} \xrightarrow{\text{LLC}^+} \{\varphi \text{ s.t. } L\varphi / \mathbb{Q}_p \text{ is wildly ramified}\}$$

least understood

What reps do these correspond to?

G has "Moy-Prasad filtrations" $(\mathbb{Z}_p \supseteq p\mathbb{Z}_p \supseteq p^2\mathbb{Z}_p \supseteq \dots)$

ex

$$\underbrace{SL_2(\mathbb{Z}_p)}_{P} \geq \underbrace{\begin{pmatrix} 1+p\mathbb{Z}_p & p\mathbb{Z}_p \\ p\mathbb{Z}_p & 1+p\mathbb{Z}_p \end{pmatrix}}_{P_1} \geq \underbrace{\begin{pmatrix} 1+p^2\mathbb{Z}_p & p^2\mathbb{Z}_p \\ p^2\mathbb{Z}_p & 1+p^2\mathbb{Z}_p \end{pmatrix}}_{P_2} \geq \cdots$$

"parahoric" .....

Given a smooth irrep V of G, can use M-P filtrations to define depth of V.

Idea: For which r is $V^{P_r} \neq 0$?

ex V is depth zero if $\exists$ a parahoric P s.t.

$$V^{P_1} \neq 0.$$

III. Unramified LLC

$\varphi : W \times SL_2(\mathbb{C}) \longrightarrow G^\vee$

If φ is unramified, then φ is completely determined by:

- $\varphi(\mathrm{Frob}) = s$ semisimple ele. of $G^\vee$
- $\varphi\left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right) = u$ unipotent ele of $G^\vee$

s.t. $su = us$

What's known?

- all unipotent irreps have been constructed.
- A potential LLC_{un} has been defined.

Still need to check desired properties of the LLC.

How? $P = SL_2(\mathbb{Z}_p)$ $\quad P/P_1 \cong SL_2(\mathbb{F}_p)$

P_1

IV. Positive-depth representations

If $p >> 0$, then we know how to get all irreps of G.

If G is classical and $p \neq 2$

But if p is small and G is exceptional, we don't.

Suppose φ is a LP. What are the possibilities for $\varphi(W)$?

ex $G = SL_2(\mathbb{Q}_p) \Rightarrow G^\vee = PGL_2(\mathbb{C}) \simeq SO_3(\mathbb{C})$.

Possibilities for $L_\varphi / (L_\varphi)^{tr}$?

The finite subgroups of $SO_3(\mathbb{C})$ are:

- the cyclic groups
- the dihedral groups
- A_4
- S_4
- A_5